ELASTIC WAVES IN CRYSTALS AND ANISOTROPIC MEDIA

V. S. Budaev

UDC 539.3.534.231.1

The results of investigation of propagation of elastic waves in anisotropic media are discussed taking into account the two-dimensional problem of a source in an infinite medium and the Lamb problem for a half-plane. The media considered in the investigation are those for which the equations of motion under plane deformation conditions are characterized by four constants.

1. For a number of anisotropic media the equations of motion under plane deformation conditions are written in the form [1, 2]

$$\begin{aligned} c_1\frac{\partial^2 u}{\partial x^2}+c_2\frac{\partial^2 w}{\partial x\,\partial z}+c_3\frac{\partial^2 u}{\partial z^2}-\rho\frac{\partial^2 u}{\partial t^2}&=-\rho a_1 f \\ c_3\frac{\partial^2 w}{\partial x^2}+c_2\frac{\partial^2 u}{\partial x\,\partial z}+c_4\frac{\partial^2 w}{\partial z^2}-\rho\frac{\partial^2 w}{\partial t^2}&=-\rho a_2 f \end{aligned} \tag{1.1}$$

Here u, w are the components of the displacements along x and z axes, ρ is the density, t is the time, a_1, a_2 are constants, and f is some function of x, z, t. The constants c_1, c_2, c_3, c_4 are expressed in terms of the elastic constants of the medium.

In the case of cubic crystals we have

$$c_1=c_4=a_{11},\ c_2=a_{12}+a_{44},\ c_3=a_{44}$$

In the case of hexagonal crystals, certain types of rhombohedral crystals, and transversally isotropic media the coefficients in the equation are expressed in terms of the elastic constants in the form

$$c_1=a_{11},\ c_2=a_{13}+a_{44},\ c_3=a_{44},\ c_4=a_{33}$$

In the case of transversally isotropic media the z axis is perpendicular to the plane of isotropy.

We assume that the coefficients of the equation satisfy the conditions of rigorous hyperbolicity and the conditions of positive-definiteness of the elastic energy. In notation α, β, γ, where

$$\alpha=c_3/c_1,\ \beta=c_3/c_4,\ \gamma=1+\alpha\beta-c_2^2/c_1c_4$$

the conditions of strict hyperbolicity are written in the form

$$-2\sqrt{\alpha\beta}<\gamma<1+\alpha\beta \tag{1.2}$$

In the case of cubic crystals the conditions of positive-definiteness of the elastic energy are of the form [3]

$$a_{11}>0,\ a_{44}>|a_{12}|,\ a_{11}+2a_{12}>0 \tag{1.3}$$

In the case of hexagonal crystals and transversally isotropic media we have [3]

$$a_{11}>0,\ a_{11}>|a_{12}|,\ (a_{11}+a_{12})\,a_{33}>2a_{13}^2 \tag{1.4}$$

We divide the entire ensemble of anisotropic media into groups in the following way:

1) media whose elastic constants satisfy the conditions [1, 2]

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 143-149, November-December, 1974. Original article submitted April 8, 1974.

$$\begin{aligned}&\gamma^2 \geqslant 4\alpha\beta,\ \gamma > \alpha(\beta+1),\ \gamma > \beta(1+\alpha)\\&[2\beta(1+\alpha)-\gamma(1+\beta)] \geqslant -|\beta-1|\sqrt{\gamma^2-4\alpha\beta}\\&[2\alpha(1+\beta)-\gamma(1+\gamma)] \geqslant -|\alpha-1|\sqrt{\gamma^2-4\alpha\beta}\\&0<\alpha<1,\ 0<\beta<1\end{aligned} \tag{1.5}$$

2) media for which

$$\alpha\beta > 1,\ \gamma^2 \geqslant 4\alpha\beta \tag{1.6}$$

3) the rest, in particular, media for which

$$\gamma^2 < 4\alpha\beta$$

The first group is quite extensive. It includes many minerals, for example, rock salt, sylvite, feldspar, beryl, sandstone, and also ice and a number of crystals of pure metals having densely packed hexagonal structure. It also includes all isotropic media, which we obtain by putting $\alpha = \beta$, $\gamma = 2\alpha$, in addition to (1.5).

The media of the third group are also abundant. This group includes crystals of pure metals having cubic array (gold, silver, copper, iron, potassium, lithium, sodium, lead, bismuth, tungsten, etc.) and also crystals of certain metals with densely packed hexagonal structure (zinc, beryllium).

There are no examples for media of the second group. We shall show below that media of this group do not exist.

2. The determination of the fundamental solution of system (1.1) amounts to solving (1.1) in which $f = \delta(x)\delta(z)\delta(t)$. Here $\delta(x)$, $\delta(z)$, $\delta(t)$ are Dirac delta functions. The fundamental solution determines the displacement field excited in the medium by a concentrated pulsed perturbation source. For the media of the first group the solution of the problem is given in [1]. For points on the x axis the solution is written in the form

$$\begin{aligned}u &= -\frac{\rho a_1}{4\pi t c_0}\sum_{n=1}^{2}\left\{\frac{T_n^\circ(\varepsilon)}{\omega_n(\varepsilon)K(\varepsilon)}H(t-t_n)\right\}\\ w &= \frac{\rho a_2}{4\pi t c_0}\sum_{n=1}^{2}\{[\omega_n(\varepsilon)T_n^\circ(\varepsilon)/K(\varepsilon)]H(t-t_n)\}\end{aligned} \tag{2.1}$$

$$\begin{aligned}\omega_n(\varepsilon) &= \frac{c_2 q_n(\varepsilon)}{c_4 q_n^2(\varepsilon)-c_3+\rho\varepsilon^2} = -\frac{c_3 q_n^2(\varepsilon)-c_1+\rho\varepsilon^2}{c_2 q_n(\varepsilon)}\\ c_0 &= c_1c_3/c_2,\ T_n^\circ(\varepsilon) = (-1)^{n+1}T_n(\varepsilon),\ \varepsilon = x/t\end{aligned} \tag{2.2}$$

The values of the remaining quantities are given in [1]. The wave velocities in the direction of the x axis are

$$c_a = \sqrt{c_1/\rho},\ c_b = \sqrt{c_3/\rho},\ c_a > c_b$$

Here c_a denotes the velocity of the quasilongitudinal wave, and c_b denotes the velocity of the quasitransverse wave. The quantities $q_n = q_n(\varepsilon)$ are related to the roots of the characteristic equation of the system $\mu_n = \mu_n(\theta)$ through the formula

$$\mu_n(\theta) = i\theta q_n(\varepsilon),\ \theta = 1/\varepsilon,\ n = 1, 2$$

The functions $\mu_n(\theta)$ determine the refraction surfaces (curves). The functions q_n are given by the following expressions:

$$\begin{aligned}&q_1 = (-M_1+\sqrt{M_1^2-M_2})^{1/2},\ q_2 = (-M_1-\sqrt{M_1^2-M_2})^{1/2}\\&M_1 = (2c_3c_4)^{-1}[c_2^2+c_3(\rho\varepsilon^2-c_3)+c_4(\rho\varepsilon^2-c_1)]\\&M_2 = (\rho\varepsilon^2-c_3)(\rho\varepsilon^2-c_1)/c_3c_4\end{aligned} \tag{2.3}$$

The arrangement of the branch points of the radicals in (2.3), the form of functions $\mu_n(\theta)$ on the real θ axis, and the geometry of the wavefronts for the media of the first group are discussed in [1, 2]; for the media of the third group they are discussed in [4], and the essential differences between the media of the first and third group are indicated.

We consider the solution near the wavefronts intersecting the x axis at the points $x_a = c_a t$, $x_b = c_b t$, respectively.

From (2.3) and (2.2) we obtain

$$\lim_{\varepsilon\to c_a} q_1(\varepsilon) = 0, \quad \lim_{\varepsilon\to c_a} \omega_1(\varepsilon) = \lim_{q_1\to 0} [c_2/(c_1 - c_3)]\, q_1(\varepsilon)$$

Let $\rho\varepsilon^2 = c_1 - \delta$, $\delta > 0$, $\delta \to 0$. From (2.2) we obtain the following equation for q_1:

$$(c_3 q_1^2 - \delta)\,[c_4 q_1^2 + (c_1 - c_3) - \delta] + c_2^2 q_1^2 = 0 \tag{2.4}$$

For $\delta \to 0$ we have

$$\rho\varepsilon^2 \to c_1, \ \varepsilon \to c_a, \ q_1^2 \to 0$$

Neglecting the quantity q_1^4 compared to q_1^2 in (2.4), we obtain the following expression with an accuracy up to δ:

$$q_1^2 = A\delta, \quad A \equiv \frac{c_1 - c_3}{c_3(c_1 - c_3) + c_2^2}.$$

For the investigated case of the media of the first group $c_1 > c_3$, i.e.,

$$(c_1 - c_3) > 0, \ c_3(c_1 - c_3) + c_2^2 > 0$$

For values of ε close to c_α we obtain

$$q_1^2 \approx A\delta \equiv A_1\sigma, \ \sigma = \delta/\rho, \ A_1 \equiv A\rho$$
$$\delta = c_1 - \rho\varepsilon^2, \ \sigma t^2 = (c_a t - \varepsilon t)(c_a t + \varepsilon t)$$

For $\varepsilon \to c_\alpha$ or $x \to x_\alpha$ we have

$$q_1(\varepsilon) \to B(x_a - x)^{1/2}$$

where $x_\alpha = c_\alpha t$, x is the instantaneous coordinate, $x = \varepsilon t$.

Considering that $T_1(\varepsilon)$, $K(\varepsilon)$ are finite and nonzero for $\varepsilon = c_\alpha$, for u and w we obtain

$$u(x, 0, t) \to C(x_a - x)^{-1/2}, \quad w(x, 0, t) \to D(x_a - x)^{1/2}, \ x \to x_a$$

where C, D, are independent of Δx, $\Delta x = x_\alpha - x$.

Similarly, in approaching the front of the quasitransverse wave we find that $w(x, 0, t)$ increases as $(x_b - x)^{-1/2}$. At the point $x = x_b$ $u(x, 0, t)$ is finite and nonzero.

Near the wavefronts on the x axis the solution in the case of anisotropic media has the same behavior as in the case of isotropic media. It can be shown that on approaching the wavefront along any ray passing through the point $x = 0$, $z = 0$ the solution has similar behavior. This result is valid also for the media of the third group. Near the boundaries of lacunas the solution behaves as near the wavefronts.

For all media of the first group, including isotropic media, the qualitative behavior of the displacement curves is the same for the points on the coordinate axes. Examples of the computation are shown in [1]. For media in which c_1 and c_4 have close values, we find that the closer the value of the coefficient $\Delta_A = (c_1 - c_3)/c_2$ gets to unity, the smaller is the difference between the solution at the coordinate axes for this anisotropic medium of the first group and the solution for the isotropic medium with the same values of coefficients c_1 and c_3, i.e., with the same velocities of propagation of longitudinal and transverse waves equal to c_α and c_b.

In the case of media of the third group a large diversity of the forms of the displacement curves is observed for points on the coordinate axes and on rays passing through the point of application of the stress depending on whether functions $\mu_n(\theta)$ and $\theta_n(\mu)$ belong to the second or third type (according to the classification in [4]) and on the values of α, β compared to unity.

3. Let us consider the Lamb problem for a half-space. For media of the first group the solution of the Lamb problem (action of a concentrated pulsed load on a half-plane) is given in [2] and for media of the third group, in [4]. The displacement curves at points of the boundaries are also given for some materials.

We consider the Rayleigh equation for each of the three groups of media and the solutions near the wavefronts and near the Rayleigh phase of the displacements.

The Rayleigh equation for an anisotropic medium is written in the form

$$R(\xi) \equiv \{[c_1c_4 - (c_2 - c_3)^2]\,\xi^2 - b^2\}\sqrt{1/c_3 - \xi^2} - \sqrt{c_1c_4}\,\sqrt{1/c_1 - \xi^2} = 0, \quad \xi^2 = 1/\rho\varepsilon^2 \qquad (3.1)$$

The function of ξ occurring in the left-hand side is called the Rayleigh function.

According to [5], the condition for the roots of Eq. (3.1) to be real is written in the form $c_1c_4 > (c_2 - c_3)^2$. In notation α, β, γ this inequality becomes

$$\gamma > 2\alpha\beta - 2\sqrt{\alpha\beta}\sqrt{1 + \alpha\beta - \gamma} \qquad (3.2)$$

Considering the condition of hyperbolicity (1.2) and the condition $\gamma^2 \geq 4\alpha\beta$, for media of the first group we find that inequality (3.2) should be considered only in the interval $2\sqrt{\alpha\beta} < \gamma < 1 + \alpha\beta$.

The fulfillment of inequality (3.2) is equivalent to the statement that for given α, β the function of γ occurring in the left side will be larger than the function in the right side for all values of γ in the investigated interval.

Let us consider the plane $f\gamma$. The function $f_1 = \gamma$ corresponds to a straight line passing through the coordinate origin at an angle $\lambda = \pi/4$ to the γ axis. The function

$$f_2 = 2\alpha\beta - 2\sqrt{\alpha\beta}\sqrt{1 + \alpha\beta - \gamma}$$

takes the following values at the ends of the interval:

$$f_2 = 4\alpha\beta - 2\sqrt{\alpha\beta} \equiv f_{21} \quad \text{for} \quad \gamma = 2\sqrt{\alpha\beta}$$
$$f_2 = 2\alpha\beta \equiv f_{22} \quad \text{for} \quad \gamma = 1 + \alpha\beta$$

In the case of media of the first group $\alpha\beta < 1$, $2\alpha\beta < 2\sqrt{\alpha\beta}$ and, hence,

$$f_{21} \equiv 2\alpha\beta + 2\alpha\beta - 2\sqrt{\alpha\beta} < 2\alpha\beta \equiv f_{22}$$

Considering that $f_2' = (\sqrt{\alpha\beta}/\sqrt{1 + \alpha\beta - \gamma}) > 0$, we find that in the interval $2\sqrt{\alpha\beta} < \gamma < 1 + \alpha\beta$ f_2 increases monotonically and lies below the straight line $f_1 = \gamma$. Therefore, inequality (3.2) is always satisfied for media of the first group, i.e., the roots of the Rayleigh equation are always positive.

In the case of media of the second group

$$f_2 = 2\sqrt{\alpha\beta} \quad \text{for} \quad \gamma = 2\sqrt{\alpha\beta}$$
$$f_2 = 2\alpha\beta \quad \text{for} \quad \gamma = 1 + \alpha\beta$$

In the interval $2\sqrt{\alpha\beta} < \gamma < 1 + \alpha\beta$ f_2 is again a monotonically increasing function, and at $\gamma = 2\sqrt{\alpha\beta}$ the values of f_1 and f_2 are equal.

Since $\alpha\beta > 1$, we have

$$1 + \alpha\beta < \alpha\beta + \alpha\beta \equiv 2\alpha\beta$$

and therefore at the point $\gamma = 1 + \alpha\beta$ f_2 is greater than f_1. Thus, either the curve for f_2 lies entirely above the straight line $f_1 = \gamma$, or they have a point of intersection, i.e., at some point $\gamma = \gamma_*$,

$$f_1(\gamma_*) = f_2(\gamma_*), \quad 2\sqrt{\alpha\beta} < \gamma_* < 1 + \alpha\beta \qquad (3.3)$$

From Eq. (3.3) we obtain $\gamma_* = 2\sqrt{\alpha\beta}$, i.e., the curves for f_1 and f_2 do not intersect anywhere except at the point $\gamma = 2\sqrt{\alpha\beta}$ lying at the end of the interval. Hence, it follows that for media of the second group inequality (3.2) is not satisfied at any point of the interval $2\sqrt{\alpha\beta} < \gamma < 1 + \alpha\beta$, i.e., in the case of media of the second group the Rayleigh equation does not have real roots. Putting $\xi = i\nu$, we find that the function $R(\xi)$ changes sign in the interval $0 < \nu < \infty$, and therefore the roots of Rayleigh equation lie on the imaginary axis of the ξ plane.

From the conditions of positive-definiteness of the elastic energy (1.3), (1.4) we obtain the inequality

$$c_1c_4 > (c_2 - c_3)^2$$

i.e., for real media the condition for the roots of the Rayleigh equation to be real are always satisfied, and therefore media of second group do not exist. The solutions of Eqs. (1.1) under conditions (1.6), (1.2) can be of interest only as some hyperbolic solutions.

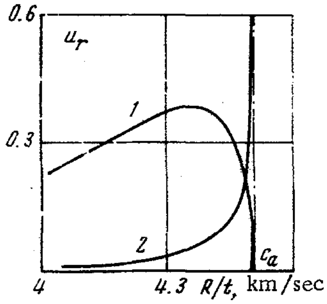

Fig. 1

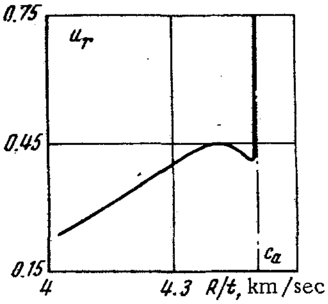

Fig. 2

If the displacement curves are plotted for the points of the free boundary in the Lamb problem, then we find that the displacements are finite everywhere, and the function w(x, 0, t) has its maximum value at the point of application of the force x = 0. In the case of real media the displacements at the points of the surface z = 0 are maximum (become infinite) for $x = c_R t$, where c_R is the velocity of the Rayleigh wave.

Following the above arguments it can be shown for media of the third group that the condition for the roots of the Rayleigh equation to be real is always satisfied.

Let us consider media of the first group. Near the leading front of the perturbation wave propagating in the half-plane $z \geq 0$ the solution of the Lamb problem is finite at the points of the free boundary z = 0 and becomes zero on the front $x_\alpha = c_\alpha t$. On approaching the perturbation wavefront along the ray passing at an angle to the x axis the radical component of the displacements increases rapidly going to infinity on the front. With the decrease of the angle (the angle formed by the ray originating from the point z = 0, x = 0 with the x axis) to zero the curves of the radial displacements for $\varphi \neq 0$ make a continuous transition to the curve u(x, 0, t).

The structure of the perturbation wave front at the interior points of the half-plane (in particular, passing of the radial component of the displacements to infinity) is determined by the derivative $\partial\theta_1/\partial t$ where θ_1 is the root of the equation $t - \theta_1 x - \mu_1(\theta_1)z = 0$. This derivative occurs in the solution as a factor [2, 4].

In the case of isotropic media we have

$$\frac{\partial\theta_1}{\partial t} = \frac{1}{R}\left(\cos\varphi - i\frac{\sin\varphi}{\sqrt{t^2 - a^2R^2}}\right) \equiv H_1 - iH_2$$

$$R^2 = x^2 + z^2,\ x = R\cos\varphi,\ z = R\sin\varphi,\ a = \sqrt{\rho/c_1}$$

The growth of the radial component to infinity on approaching the wave front ($t = \alpha R$) is caused by the component H_2. With the increase of φ the contribution of H_2 increases and decreases rapidly when φ tends to zero, becoming zero for $\varphi = 0$.

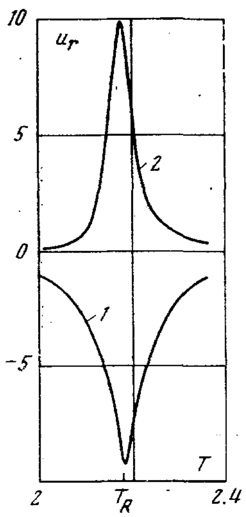

Fig. 3

The qualitative behavior of the contributions to the radial component of the displacements near the wavefront coming from the terms H_1 (curve 1) and H_2 (curve 2) (total dependence for angle $\varphi = 2°$ is shown in Fig. 2) is shown in Fig. 1 for $\varphi = 2°$ for the case of an isotropic medium with the velocities of propagation of the waves equal to c_α = 4500 m/sec, c_b = 2500 m/sec.

For large values of angle φ the radial component of the displacements near the wavefront ($t = \alpha R$) has the form of curve 2 in Fig. 1, since the contribution from H_2 predominates. For $\varphi = 0$, i.e., at the points of the surface z = 0, the radial component [equal to u(x, 0, t)] near the point $x_\alpha = c_\alpha t$ has the same behavior as curve 1 in Fig. 1, since the contribution of H_2 is zero.

These results are valid also for the solution near the leading front of the perturbation wave in all media of the first group. Just as in the case of isotropic media on approaching the perturbation wave front the radial and tangential components vary proportional to $(R_\alpha - R)^{-1/2}$ or $(R_\alpha - R)^{1/2}$, respectively, where R_α is the position of the wavefront, and R is the instantaneous coordinate (for φ = const).

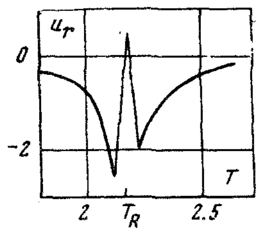

Fig. 4

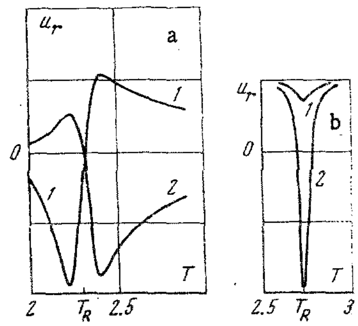

Fig. 5

In the case of media of the third group the solution on the front vanishes at the points of the surface. The behavior of the displacement curve at the surface points or at the interior points of the half-plane may be very different from the media of the first group. Some results for the surface points are given in [4].

Let us consider the characteristics of the form of the radial displacement curves for media of the first and third groups at the interior points of the media taking the dependence $u_r = u_r(T)$ as an example, where u_r is the radial component of the displacements, and T is nondimensional time, $T = t/aR$. The dependence $u_r(T)$ represents seismograms of radial displacements for points lying on rays passing at an angle φ to the x axis. In the vicinity of $T = T_R$, where T_R is the time of arrival of the Rayleigh displacement phase at the given point, both quasilongitudinal and quasitransverse waves contribute to the displacements.

The contribution of each wave may change significantly on passing over from one medium to another.

The displacement curves for each wave are shown in Fig. 3 (curve 1 — quasilongitudinal wave, curve 2 — quasitransverse wave), indicating the contribution from each of the waves to the total displacement. The curves are plotted for an isotropic medium (c_a = 6260 m/sec, c_b = 3080 m/sec) in the vicinity of the point $T = T_R$ for points lying on the ray passing at an angle $\varphi = 2°$. The form of the curves is typical for media of the first group; the relation between them is retained, since in the vicinity of $T = T_R$ for $\varphi = 2°$ the total pattern for media of the first group has the form shown in Fig. 4 (quantitative data are not given in the figure, since they are not of interest).

The curves showing the contribution of each wave to the magnitude of the radial displacements in the vicinity of $T = T_R$ at the points on the ray with $\varphi = 2°$ are shown in Fig. 5a for the case of zinc (third group of media, $\alpha < 1$, $\beta < 1$; curve 1 — quasilongitudinal wave, curve 2 — quasitransverse wave).

Similar curves for a model anisotropic medium (third group, $\alpha = 3$, $\beta = 0.25$, $\gamma = 1.74$) are shown in Fig. 5b.

As seen from the curves in Fig. 5, in the case of media of the third group the nature of the displacements at the interior points of the medium can be very different from the first group.

The author thanks S. A. Khristianovich and E. I. Shemyakin for attention to the work.

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